CHARACTERISTICS OF A TRUE PARAMETER OF A HIDDEN MARKOV MODEL

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Abstract. Representation which generates the observed process of a hidden Markov model is not unique. The simplest one, that is, the one with minimum size is called a true parameter. This article is aimed to present characteristics of this parameter.
Key words: Hidden Markov, representations, true parameter.

1. Introduction

According to [3], representation for a hidden Markov model is not unique. Our main interest is to find the *simplest* one, that is, the one with *minimum size*. Such representation will be called a *true parameter*. Our task is to identify a true parameter and its size. Therefore, the main aim of this article is to collect facts concerning the true parameter.

For this purpose, we begin with definition of a hidden Markov model, representations and equivalent representations in the first section. The second section will present definion of a true parameter of a hidden Markov model and its characteristics.

2. A hidden Markov model and its representations

Let $\{X_t : t \in \boldsymbol{N}\}$ be a finite state Markov chain defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $\{X_t\}$ is not observed directly, but rather there is an *observation* process $\{Y_t : t \in \boldsymbol{N}\}$ defined on $(\Omega, \mathcal{F}, P)$. Then consequently, the Markov chain is said to be *hidden* in the observations. A pair of stochastic processes $\{(X_t, Y_t) : t \in \boldsymbol{N}\}$ is called a hidden Markov model. Precisely, according to [1], a hidden Markov model is formally defined as follows.

Definition 2.1. A pair of discrete time stochastic processes $\{(X_t, Y_t) : t \in \boldsymbol{N}\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a set $\boldsymbol{S} \times \mathcal{Y}$, is said to be a ***hidden Markov model*** (HMM), if it satisfies the following conditions.

1. $\{X_t\}$ is a finite state Markov chain.
2. Given $\{X_t\}$, $\{Y_t\}$ is a sequence of conditionally independent random variables.
3. The conditional distribution of Y_n depends on $\{X_t\}$ only through X_n.
4. The conditional distribution of Y_t given X_t does not depend on t.

Assume that the Markov chain $\{X_t\}$ ***is not observable***. The cardinality K of $\boldsymbol{S}$, will be called the ***size*** of the hidden Markov model.

Since the Markov chain $\{X_t\}$ in a hidden Markov model $\{(X_t, Y_t)\}$ is not observable, then inference concerning the hidden Markov model has to be based on the information of $\{Y_t\}$ alone. By knowing the finite dimensional joint distributions of $\{Y_t\}$, parameters which characterize the hidden Markov model can then be analysed.

From [3], it can be seen that the law of the hidden Markov model $\{(X_t, Y_t)\}$ is completely specified by :
(a). The size K.
(b). The transition probability matrix $A = (\alpha_{ij})$, satisfying

$$\alpha_{ij} \geq 0 \qquad , \sum_{j=1}^{K} \alpha_{ij} = 1, \qquad i, j = 1, \ldots, K.$$

(c). The initial probability distribution $\pi = (\pi_i)$ satisfying

$$\pi_i \geq 0, \qquad i = 1, \ldots, K, \qquad \sum_{i=1}^{K} \pi_i = 1.$$

(d). The vector $\theta = (\theta_i)^T$, $\theta_i \in \Theta$, $i = 1, \ldots, K$, which desribes the conditional
densities of Y_t given $X_t = i$, $i = 1, \ldots, K$.

Definition 2.2. Let

$$\phi = (K, A, \pi, \theta).$$

The parameter ϕ is called a ***representation*** of the hidden Markov model
$\{(X_t, Y_t)\}$.

Thus, the hidden Markov model $\{(X_t, Y_t)\}$ can be represented by a representation $\phi = (K, A, \pi, \theta)$.

On the otherhand, we can also generate a hidden Markov model $\{(X_t, Y_t)\}$ from a representation $\phi = (K, A, \pi, \theta)$, by choosing a Markov chain $\{X_t\}$ which takes values on $\{1, \ldots, K\}$ and its law is determined by the $K \times K$-transition probability matrix A and the initial probability π, and an observation process $\{Y_t\}$ taking values on $\mathcal{Y}$, where the density functions of Y_t given $X_t = i$, $i = 1, \ldots, K$ are determined by θ.

Let $\phi = (K, A, \pi, \theta)$ and $\widehat{\phi} = (\widehat{K}, \widehat{A}, \widehat{\pi}, \widehat{\theta})$ be two representations which respectively generate hidden Markov models $\{(X_t, Y_t)\}$ and $\{(\widehat{X}_t, Y_t)\}$. The $\{(X_t, Y_t)\}$ takes values on $\{1, \ldots, K\} \times \mathcal{Y}$ and $\{(\widehat{X}_t, Y_t)\}$ takes values on $\{1, \ldots, \widehat{K}\} \times \mathcal{Y}$. For any $n \in \boldsymbol{N}$, let $p_\phi(\cdot, \cdots, \cdot)$ and $p_{\widehat{\phi}}(\cdot, \cdots, \cdot)$ be the n-dimensional joint density function of $Y_1, \ldots Y_n$ with respect to ϕ and $\widehat{\phi}$. Suppose that for every $n \in \boldsymbol{N}$,

$$p_\phi(Y_1, \ldots, Y_n) = p_{\widehat{\phi}}(Y_1, \ldots, Y_n).$$

Then $\{Y_t\}$ has the same law under ϕ and $\widehat{\phi}$. Since in hidden Markov models $\{(X_t, Y_t)\}$ and $\{(\widehat{X}_t, Y_t)\}$, the Markov chains $\{X_t\}$ and $\{\widehat{X}_t\}$ are not observable and we only observed the values of $\{Y_t\}$, then theoretically, the hidden Markov models $\{(X_t, Y_t)\}$ and $\{(\widehat{X}_t, Y_t)\}$ are *indistinguishable*. In this case, it is said that $\{(X_t, Y_t)\}$ and $\{(\widehat{X}_t, Y_t)\}$ are *equivalent*. The representations ϕ and $\widehat{\phi}$ are also said to be *equivalent*, and will be denoted as $\phi \sim \widehat{\phi}$.

For each $K \in \boldsymbol{N}$, define

$$\begin{aligned} \Phi_K = \Big\{ \phi : \phi = (K, A, \pi, \theta), \text{ where } A, \pi \text{ and } \theta \text{ satisfy :} \\ A = (\alpha_{ij}), \quad \alpha_{ij} \geq 0, \quad \sum_{j=1}^{K} \alpha_{ij} = 1, \quad i, j = 1, \ldots, K \\ \pi = (\pi_i), \quad \pi_i \geq 0, \quad i = 1, \ldots, K, \quad \sum_{i=1}^{K} \pi_i = 1 \\ \theta = (\theta_i)^T, \quad \theta_i \in \Theta, \quad i = 1, \ldots, K \Big\} \end{aligned} \tag{2.1}$$

and

$$\Phi = \bigcup_{K \in \boldsymbol{N}} \phi_K. \tag{2.2}$$

The relation $\sim$ is now defined on Φ as follows.

Definition 2.3. Let $\phi, \widehat{\phi} \in \Phi$. Representations ϕ and $\widehat{\phi}$ are said to be ***equivalent***, denoted as

$$\phi \sim \widehat{\phi}$$

if and only if for every $n \in \boldsymbol{N}$,

$$p_\phi(Y_1, Y_2, \ldots, Y_n) = p_{\widehat{\phi}}(Y_1, Y_2, \ldots, Y_n).$$

Remarks 2.4. It is clear that relation $\sim$ forms an equivalence relation on Φ.

Let $\phi = (K, A, \pi, \theta) \in \Phi_K$, then under ϕ, $Y_1, \dots, Y_n$, for any n, has joint density

$$p_\phi(y_1, \dots, y_n) = \sum_{x_1=1}^{K} \cdots \sum_{x_n=1}^{K} \pi_{x_1} f(y_1, \theta_{x_1}) \cdot \prod_{t=2}^{n} \alpha_{x_{t-1}, x_t} f(y_t, \theta_{x_t}). \quad (2.3)$$

Let σ be any permutation of $\{1, 2, \dots, K\}$. Define

$$\begin{aligned} \sigma(A) &= (\alpha_{\sigma(i),\sigma(j)}) \\ \sigma(\pi) &= (\pi_{\sigma(i)}) \\ \sigma(\theta) &= (\theta_{\sigma(i)})^T. \end{aligned}$$

Let

$$\sigma(\phi) = (K, \sigma(A), \sigma(\pi), \sigma(\theta)),$$

then $\sigma(\phi) \in \Phi_K$ and easy to see from (2.3) that

$$p_\phi(y_1, \dots, y_n) = p_{\sigma(\phi)}(y_1, \dots, y_n).$$

implying $\phi \sim \sigma(\phi)$. So we have the following lemma.

Lemma 2.5. *Let $\phi \in \Phi_K$, then for every permutation σ of $\{1, 2, \dots, K\}$,*

$$\sigma(\phi) \sim \phi.$$

from [3], we have the following lemmas.

Lemma 2.6. *Let $\phi = (K, A, \pi, \theta) \in \Phi_K$, where π is a stationary probability distribution of A. Let N be the number of non-zero π_i. Then there is $\widehat{\phi} = (N, \widehat{A}, \widehat{\pi}, \widehat{\theta}) \in \Phi_N$, such that :*

1. *$\widehat{\pi}_i > 0$, for $i = 1, \dots, N$.*
2. *$\widehat{\pi}$ is a stationary probability distribution of $\widehat{A}$.*
3. *$\phi \sim \widehat{\phi}$.*

Lemma 2.7. *For any $K \in \boldsymbol{N}$ and $\phi \in \Phi_K$, there is $\widehat{\phi} \in \Phi_{K+1}$, such that $\phi \sim \widehat{\phi}$.*

By Lemma 2.7, we can define an order $\prec$ in $\{\Phi_K\}$.

Definition 2.8. Define an ***order*** $\prec$ on $\{\Phi_K\}$ by

$$\Phi_K \prec \Phi_L, \qquad K, L \in \boldsymbol{N},$$

if and only if for every $\phi \in \Phi_K$, there is $\widehat{\phi} \in \Phi_L$ such that $\phi \sim \widehat{\phi}$.

As a consequence of Lemma 2.7, Lemma 2.9 follows.

Lemma 2.9. *For every $K \in \boldsymbol{N}$,*

$$\Phi_K \prec \Phi_{K+1}.$$

From Lemma 2.9, the families of hidden Markov models represented by $\{\Phi_K\}$ are ***nested families***.

3. A true parameter and its characteristics

We begin this section with a formal definition of a true parameter.

Definition 3.1. Let $\{(X_t, Y_t)\}$ be a hidden Markov model with representation $\phi \in \Phi$. A representation $\phi^o = (K^o, A^o, \pi^o, \theta^o) \in \Phi$, is called a ***a true parameter*** of the hidden Markov model $\{(X_t, Y_t)\}$ if and only if

1. $\phi^o \sim \phi$.
2. K^o is ***minimum***, that is, there is no $\widehat{\phi} \in \Phi_K$, with $K < K^o$, such that $\widehat{\phi} \sim \phi^o$.

A true parameter $\phi^o = (K^o, A^o, \pi^o, \theta^o)$ of a hidden Markov model $\{(X_t, Y_t)\}$ is not unique, by Lemma 2.5, for every permutation σ of $\{1, \dots, K^o\}$,

$$\sigma(\phi^o) \sim \phi^o.$$

So $\sigma(\phi^o)$ is also a true parameter of the hidden Markov model $\{(X_t, Y_t)\}$.

As a straight consequence of Definition 3.1, we have the following lemma.

Lemma 3.2. *Let $\phi^o = (K^o, A^o, \pi^o, \theta^o)$ be a true parameter of a hidden Markov model $\{(X_t, Y_t)\}$. Then there is no $\phi \in \Phi_K$, with $K < K^o$ such that $\phi \sim \phi^o$.*

The next two lemmas show some properties of true parameter which generates a stationary hidden Markov model.

Lemma 3.3. *Let $\phi^o = (K^o, A^o, \pi^o, \theta^o)$ be a true parameter of a hidden Markov model $\{(X_t, Y_t)\}$. If π^o is a stationary probability distribution of A^o, then*

$$\pi_i^o > 0, \qquad \text{for } i = 1, \dots, K^o.$$

Proof :
Let N^o be the number of non-zero π_i^o's, then $1 \le N^o \le K$. If $N^o < K^o$, then by Lemma 2.6, there is $\phi = (N^o, A, \pi, \theta) \in \Phi_{N^o}$, such that $\phi \sim \phi^o$, contradicting with Lemma 3.2. Thus, it must be $N^o = K^o$. ■

Lemma 3.4. *Let $\phi^o = (K^o, A^o, \pi^o, \theta^o)$ be a true parameter of a hidden Markov model $\{(X_t, Y_t)\}$, where π^o is a stationary probability distribution of A^o. Let $\phi = (K, A, \pi, \theta) \in \Phi_K$, where $\phi \sim \phi^o$ and N be the number of non-zero π_i.*

1. *If $K = K^o$, then $N = K^o$.*
2. *If $K > K^o$, then $N \geq K^o$.*

Proof :
Let $\phi = (K, A, \pi, \theta) \in \Phi_K$, where $\phi \sim \phi^o$. By Lemma 3.2,

$$K \geq K^o.$$

Let N be the number of non-zero π_i, then

$$1 \leq N \leq K.$$

Suppose that $N < K^o$, since $\phi \sim \phi^o$, then π is a stationary probability distribution of A. By Lemma 2.6, there is $\widehat{\phi} = (N, \widehat{A}, \widehat{\pi}, \widehat{\theta}) \in \Phi_N$, such that $\phi \sim \widehat{\phi}$, implying $\widehat{\phi} \sim \phi^o$, contradicting with Lemma 3.2. Thus, it must be

$$K^o \leq N \leq K. \tag{3.1}$$

If $K = K^o$, then by (3.1), $N = K^o$. If $K > K^o$, then $N \geq K^o$. ■

Corollary 1. let $\phi^o = (K^o, A^o, \pi^o, \theta^o)$ be a true parameter of a hidden Markov model $\{(X_t, Y_t)\}$, where π^o is a stationary probability distribution of A^o. Let $\phi = (K^o, A, \pi, \theta) \in \Phi_{K^o}$. If $\phi \sim \phi^o$, then

$$\pi_i > 0, \qquad \text{for } i = 1, \ldots, K^o.$$

Proof :
This is part (a) of Lemma 3.4.

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